Paraconsistent transition systems and their logics

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Motivation

In Artificial Intelligence and Software Engineering it is often a challenge to cope with modelling contexts in which bivalent logic is not enough, entailing the need to capture

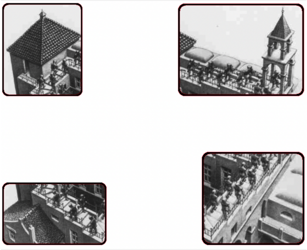
- lack of information (vagueness or uncertainty)
- excess of information (potential inconsistency)

Potentially contradictory information arrises in a number of scenarios. (e.g. knowledge representation, data integration, etc.)

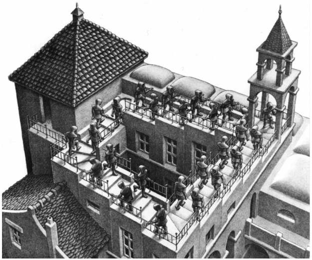
Motivation

and even on articulating different perspectives on complex information:

local consistency

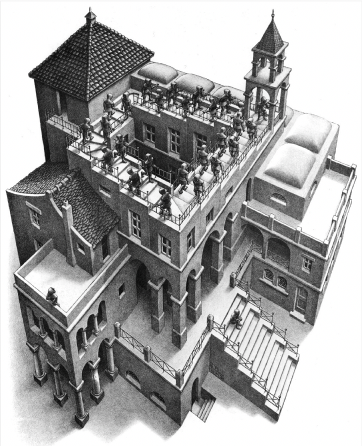


global inconsistency



Motivation

Information (reality?) is not what we thought it was ...



M. C. Escher, *Ascending and descending*

Two such scenarios

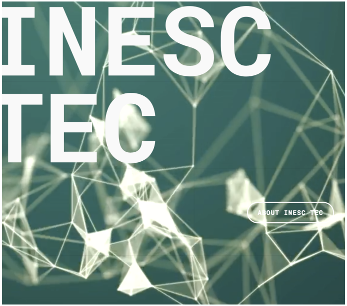
Analysis of social networks dynamics to model citizen participation in the context of the United Nations University

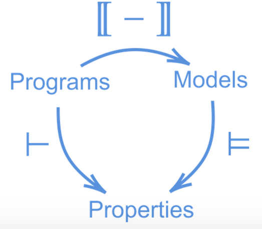
UNU
EGOV

Operational Unit on Policy-driven Digital Governance, Guimarães, Portugal, since 2014

Two such scenarios

Logics for quantum computing
in the context of the High-Assurance Software Lab INESC TEC





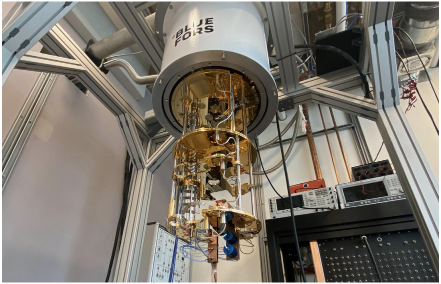
Motivation: Logics for quantum programs

... doing things right from the outset

- Logic plays in quantum computation the same essential role it plays in the classical case, namely in specification and verification
- Need to incorporate the non-classical dynamics of quantum information
  cf Smets & Baltag's work on dynamic logics for reasoning about (concrete) quantum programs e.g.
  - LQP: The dynamic logic of quantum information (2006)
    – the quantum 'version' of propositional dynamic logic
  - PLQP & Company: decidable logics for quantum algorithms (2014)
    – with a probability modality to capture the success of a test, allowing to go beyond 'qualitative' properties.

Motivation: Logics for NISQ quantum programs

but current NISQ programs put a different challenge ...



Noisy Intermediate-Scale Quantum

- Everything has a limited fidelity
- Qubits have finite coherence time
- Errors can vary across the machine
- Errors can vary across time
- Errors are correlated, non-markovian
- Errors are not always well understood

Noisy **Intermediate-Scale** Quantum

- 50—100 qubits
- Enough to be hard to simulate
- Not enough for error correction

Motivation: Logics for NISQ quantum programs

but current NISQ programs put a different challenge ...

- Need to deal with quibit coherence/decoherence times,
- and noisy gates

Combining gate inherent noise with coherence/decoherence times of the qubits upon which it acts at a particular computation stage, one gets, for each gate in the circuit, experimentally computed pairs consisting of

- a possible coherence weight
- a possible decoherence weight

which are not 'complementary' in any sense.

Somehow both these values have to be taken into consideration to compute the quality of the computation.

A role for paraconsistency?

... entails the need to deal with contradictory information ...

Is there a role for paraconsistency?

- Paraconsistent logic treats inconsistent information as potentially informative
  (originally developed in Latin America in 50's, mainly by F. Asenjo and Newton da Costa)
- Applications to quantum mechanics and quantum information theory
  (cf work from D. Chiara and W. Carnielli)

A role for paraconsistency?

Agenda

- Paraconsistent structures:
  introducing a semantics over a iMTL-algebra and a modal logic
- Labelled paraconsistent transition systems:
  introducing a category, compositional operators, bisimulation and a multimodal, process logic

PART 1

PARACONSISTENT STRUCTURES

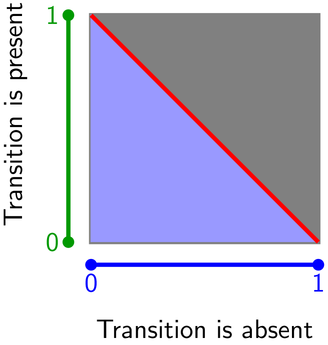
and their modal logic

Paraconsistent structures

The approach

- Transition systems with both positive and negative accessibility relations, with non complementary weights:
  - one weighting the possibility of a transition to be present (e.g. the state remaining coherent),
  - the other weighting the possibility of being absent (i.e. becoming unstable)
- used as Kripke frames for a modal logic

The structure



- Paraconsistency:
  regards inconsistent information as potentially informative.

How is information weighted?

Underlying semantic structure

- A residuated lattice, i.e. a bounded lattice $\langle A, \sqcap, \sqcup, 1, 0, \rangle$ equipped with a monoid $\langle A, \odot, e \rangle$ such that $\odot$ has a right adjoint,

$$a \odot b \leq c \;\Leftrightarrow\; b \leq a \rightharpoonup c$$

- st the monoidal operation coincides with meet, $\odot$ is $\sqcap$
- plus a prelinearity condition:

$$(a \rightharpoonup b) \sqcup (b \rightharpoonup a) = 1$$

(iMTL-algebra, after *integral monoidal t-norm based logic*)

Some examples

$\mathbf{3} = \langle \{\bot, u, \top\}, \wedge_3, \vee_3, \top, \bot, \rightarrow_3 \rangle$

$\vee_3$	$\bot$	u	$\top$
$\bot$	$\bot$	u	$\top$
u	u	u	$\top$
$\top$	$\top$	$\top$	$\top$

$\wedge_3$	$\bot$	u	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
u	$\bot$	u	u
$\top$	$\bot$	u	$\top$

$\rightarrow_3$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	$\bot$	$\top$	$\top$
$\top$	$\bot$	u	$\top$

$\ddot{\mathbf{G}} = \langle 0..1, min, max, 0, 1, \rightarrow \rangle$ (Gödel)

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{otherwise} \end{cases}$$

Properties

Some properties

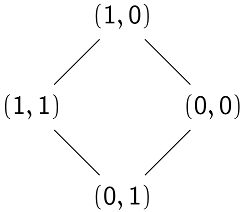
$$a \rightharpoonup (b \rightharpoonup c) = (b \sqcap a) \rightharpoonup c$$

$$b \rightharpoonup (\bigsqcap_i a_i) = \bigsqcap_i (b \rightharpoonup a_i)$$

$$(\bigsqcup_i a_i) \rightharpoonup b = \bigsqcap_i (a_i \rightharpoonup b)$$

$$b \rightharpoonup (\bigsqcup_i a_i) = \bigsqcup_i (b \rightharpoonup a_i)$$

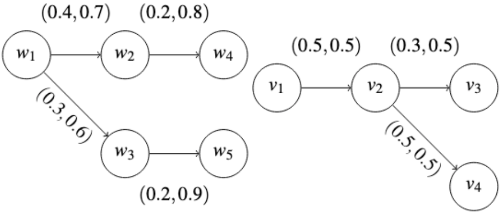
$$(\bigsqcap_i a_i) \rightharpoonup b = \bigsqcup_i (a_i \rightharpoonup b)$$



Belnap-Dunn *FOUR*

Examples

Examples over $\ddot{\mathbf{G}}$



How is information weighted?

... endowed with a metric $d : A \times A \longrightarrow \mathbb{R}^+$

Examples

$\underline{\mathbf{2}}$ and $\underline{\mathbf{3}}$

2

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

3

d	$\bot$	u	$\top$
$\bot$	0	1	2
u	1	0	1
$\top$	2	1	0

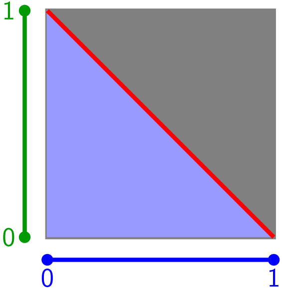
$\ddot{\mathbf{G}}$ (Gödel)

$$\ddot{\mathbf{G}} = \langle [0,1], min, max, 0, 1, \rightarrow, e \rangle$$

where $e(x,y) = \sqrt{(x-y)^2}$

How is information weighted?

(paraconsistent): $\Delta_P = \{(a,b) | D((a,b),(1,1)) \leq D((a,b),(0,0))\}$
(consistent): $\Delta_C = \{(a,b) | D((a,b),(0,0)) \leq D((a,b),(1,1))\}$
(strictly consistent): $\Delta = \Delta_P \cap \Delta_C$



Computing with pairs of weights

$\mathbf{A}^2 = \langle A \times A, \curlywedge, \curlyvee, \Longrightarrow, \neg, D \rangle$
(to compute with pairs of weights: twisted structures with a metric)

- $(a, b) \curlyvee (c, d) = (a \sqcup c, b \sqcap d)$
- $(a, b) \curlywedge (c, d) = (a \sqcap c, b \sqcup d)$
- $(a, b) \Longrightarrow (c, d) = ((a \rightharpoonup c) \sqcap (d \rightharpoonup b), a \sqcap d)$
- $\neg(a, b) = (b, a)$
- $D((a, b), (c, d)) = \sqrt{d(a, c)^2 + d(b, d)^2}$

and $\leq$ lifts from $\mathbf{A}$ generalizing Belnap-Dun *truth* order:

$$(a, b) \preccurlyeq (c, d) \text{ iff } a \leq c \text{ and } b \geq d$$

Computing with pairs of weights

The rationale for the operators is that elements of a twist-structure are regarded as classical complementary sentences

Clearly, $\Longrightarrow$ is not residuated in $\mathbf{A}^2$ as

$$(a,b) \curlywedge (c,d) \preccurlyeq (e,f) \text{ iff } (c,d) \preccurlyeq (a,b) \Longrightarrow (e,f) \text{ fails:}$$

e.g. $(a,b) = (0.8, 0.4)$, $(c,d) = (0.5, 0.2)$ and $(e,f) = (0.6, 0.3)$,

$$(a,b) \curlywedge (c,d) = (min\{0.8, 0.5\}, max\{0.4, 0.2\}) = (0.5, 0.4)$$
$$(a,b) \Longrightarrow (e,f) = (min(0.8 \to 0.6, 0.3 \to 0.4), min\{0.8, 0.3\}) = (0.6, 0.3)$$

Thus,

$$(0.5, 0.4) \preccurlyeq (0.6, 0.3) \text{ but } (0.5, 0.2) \not\preccurlyeq (0.6, 0.3)$$

Computing with pairs of weights

However, the adjunction is recovered replacing $\curlywedge$ by

$$(a, b) \otimes (c, d) = (a \sqcap c, a \rightarrow d \sqcap c \rightarrow b)$$

entailing

$$(a, b) \otimes (c, d) \preccurlyeq (e, f) \quad \text{iff} \quad (a, b) \preccurlyeq (c, d) \Rightarrow (e, f)$$

The logic $L(\mathbf{A})$

Syntax

$$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi | \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \not\Box\varphi \mid \not\Diamond\varphi \mid \circ\varphi$$

where $p \in \mathrm{Prop}$

Semantics

A model over $\mathbf{A} = \langle A, \sqcup, \sqcap, 1, 0, \rightharpoonup, d\rangle$ is a

- PTS $(W; R \subseteq W \times A \times A \times W)$ often expressed as (R^+, R^-)
- and a valuation $V : \mathrm{Prop} \times W \rightarrow A \times A$

The logic $L(\mathbf{A})$

Satisfaction

- $(w \models p) = V(p, w)$
- $(w \models \bot) = (0, 1)$
- $(w \models \neg\varphi) = \overline{\neg}(w \models \varphi)$
- $(w \models \varphi_1 \wedge \varphi_2) = (w \models \varphi_1) \overset{\wedge}{\wedge} (w \models \varphi_2)$
- $(w \models \varphi_1 \vee \varphi_2) = (w \models \varphi_1) \overset{\vee}{\vee} (w \models \varphi_2)$
- $(w \models \varphi_1 \rightarrow \varphi_2) = (w \models \varphi_1) \Longrightarrow (w \models \varphi_2)$
- $(w \models \circ\varphi) = \begin{cases} (1, 0) \text{ if } (w \models \varphi) \in \Delta_C \\ (0, 1) \text{ otherwise} \end{cases}$

The logic $L(\mathbf{A})$

Satisfaction

- $(w \models \Box\varphi) = (\boxplus(w, \varphi^+), \diamondplus(w, \varphi^-))$
- $(w \models \Diamond\varphi) = (\diamondplus(w, \varphi^+), \boxplus(w, \varphi^-))$
- $(w \models \not\Box\varphi) = (\diamondminus(w, \varphi^-), \boxminus(w, \varphi^+))$
- $(w \models \not\Diamond\varphi) = (\boxminus(w, \varphi^-), \diamondminus(w, \varphi^+))$

where

- $\boxplus(w, \varphi^*) = \bigsqcap_{w' \in R[w]}(R^+(w, w') \rightharpoonup (w' \models \varphi)^*)$
- $\boxminus(w, \varphi^*) = \bigsqcap_{w' \in R[w]}(R^-(w, w') \rightharpoonup (w' \models \varphi)^*)$
- $\diamondplus(w, \varphi^*) = \bigsqcup_{w' \in R[w]}(R^+(w, w') \sqcap (w' \models \varphi)^*)$
- $\diamondminus(w, \varphi^*) = \bigsqcup_{w' \in R[w]}(R^-(w, w') \sqcap (w' \models \varphi)^*)$

Negation and duality

Interpretation of $(w \models \neg\varphi)$ as $\bar{\neg}(w \models \varphi)$

- turns $\neg$ into an involution,
- satisfying $\neg\psi \leq \neg\phi$ if $\phi \leq \psi$ as

$$
\begin{aligned}
& \bar{\neg}(a', b') \preccurlyeq \bar{\neg}(a, b) \text{ if } (a, b) \preccurlyeq (a', b') \\
\equiv\ & (b', a') \preccurlyeq (b, a) \quad \text{if} \quad (a, b) \preccurlyeq (a', b') \\
\equiv\ & b' \leq b \text{ and } a' \geq a \quad \text{if} \quad a \leq a' \text{ and } b \geq b'
\end{aligned}
$$

Thus, De Morgan laws hold,
However,

$$(a, b) \curlywedge \bar{\neg}(a, b) \;=\; (a, b) \curlywedge (b, a) \;=\; (a \sqcap b, b \sqcup a)$$

which does not coincide with $(0, 1)$, the interpretation of $\perp$, even if, in a sense, it 'gets closer' by descending in the underlying lattice.

Modalities are dual

The two pairs of modalities quantify over the positive and the negative component of the accessibility relation.

Thus, it assigns an existential, 'diamond-like' behaviour to $\not\Box$, and dually a universal, 'box-like' behaviour to $\not\Diamond$.

$$\Box\neg\varphi \equiv \neg\Diamond\varphi \qquad \not\Box\neg\varphi \equiv \neg\not\Diamond\varphi$$
$$\Diamond\neg\varphi \equiv \neg\Box\varphi \qquad \not\Diamond\neg\varphi \equiv \neg\not\Box\varphi$$

Modalities are dual

$$(w \models \boxslash \neg \varphi)$$

$=$ { definition of $\models$ }

$$(\diamondminus(w, (\neg\varphi)^{-}), \boxminus(w, (\neg\varphi)^{+}))$$

$=$ { definition of $\boxminus$, $\diamondminus$ }

$$\left(\bigsqcup_{w' \in W} \{R^{-}(w, w') \sqcap (w' \models \neg\varphi)^{-}\}, \bigsqcap_{w' \in W} \{R^{-}(w, w') \rightharpoonup (w' \models \neg\varphi)^{+}\} \right)$$

$=$ { definition of $\models$ }

$$\left(\bigsqcup_{w' \in W} \{R^{-}(w, w') \sqcap (\overline{\neg}(w' \models \varphi))^{-}\}, \bigsqcap_{w' \in W} \{R^{-}(w, w') \rightharpoonup (\overline{\neg}(w' \models \varphi))^{+}\} \right)$$

$=$ { definition of $\overline{\neg}$ }

$$\left(\bigsqcup_{w' \in W} \{R^{-}(w, w') \sqcap (w' \models \varphi)^{+}\}, \bigsqcap_{w' \in W} \{R^{-}(w, w') \rightharpoonup (w' \models \varphi)^{-}\} \right)$$

$=$ { definition of $\diamondminus$ and $\boxminus$ }

$$(\diamondminus(w, \varphi^{+}), \boxminus(w, \varphi^{-}))$$

$=$ { definition of $\overline{\neg}$ }

$$\overline{\neg}(\boxminus(w, \varphi^{-}), \diamondminus(w, \varphi^{+})) \;=\; (w \models \neg \not\Diamond \varphi)$$

PART 2

PARACONSISTENT LABELLED TRANSITION SYSTEMS

as a concrete modelling tool

Going multimodal

Paraconsitent Labelled Transition Systems

- Introduce labels from a set of identifiers *Act* and an initial state
- Define morphism to organise PLTSinto a category
- Derive an algebra, to get new PLTS from old, from the underlying categorical structure
  (... possibly leading to a language and a dynamic logic)
- Develop a multimodal logic (*à la* Hennessy-Milner) for paraconsistent processes

Paraconsitent Labelled Transition Systems - PLTS

A PLTS over a iMTL-algebra $\boldsymbol{A}$, and a set of atomic actions Act is a structure $\langle W, i, R \rangle$ where,

- W is a non-empty set of states
- $i \in W$ is the initial state
- $R = (R_a \colon W \times W \longrightarrow A \times A)_{a \in Act}$ is an Act-indexed family of functions.

$$R_a(w_1, w_2) = (\alpha, \beta)$$

with α weighting evidence of the transition through a and β its absence.

Paraconsitent Labelled Transition Systems - PLTS

Morphism: $\langle W, i, R\rangle \longrightarrow \langle W', i', R'\rangle$

- $\sigma : W \longrightarrow W'$
- $\lambda : Act \longrightarrow_{\perp} Act'$

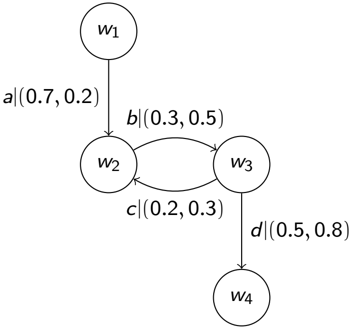
st $\sigma(i) = i'$ and for any $a \in Act$,

$$R_a(w, w') \preccurlyeq R'^{\perp}_{\lambda(a)}(\sigma(w), \sigma(w'))$$

where $R^{\perp} = R \cup R_{\perp}$ with $R_{\perp}(w, w) = (1, 0)$ for any state $w \in W$

Example

$$h = \{w_1 \mapsto v_1, w_2 \mapsto v_2, w_3 \mapsto v_3, w_4 \mapsto v_4\}$$



New PLTS from old

... exploring the underlying categorical structure of $Pt_{\mathbf{A}}$

Restriction and relabeling

... for $\lambda : Act' \to Act$ an inclusion:
are

- $T \upharpoonright \lambda$ is a Cartesian lifting in $Pt_{\mathbf{A}}$
- $T\{\lambda\}$ is a co-Cartesian lifting in $Pt_{\mathbf{A}}$

New PLTS from old

Parallel Composition

$T_1 \times T_2$ is the categorical product in $Pt_{\mathbf{A}}$:

$$T_1 \times T_2 = \langle W_1 \times W_2, (i_1, i_2), R \rangle$$

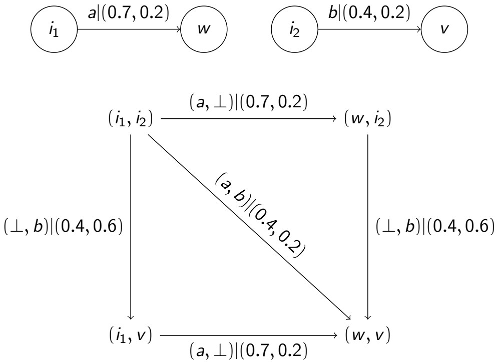
over

$$Act_1 \times_\perp Act_2 = \{(a, \perp) \mid a \in Act_1\} \cup \{(\perp, b) \mid b \in Act_2\} \cup \{(a, b) \mid a \in Act_1, b \in Act_2\}$$

such that

$$\begin{aligned} R_{(a,b)}((w_1, w_2), (v_1, v_2)) &= (\alpha, \beta) \text{ iff} \\ (R_1)_a^\perp(w_1, v_1) &= (\alpha_1, \beta_1) \text{ and} \\ (R_2)_b^\perp(w_2, v_2) &= (\alpha_2, \beta_2) \text{ and} \\ (\alpha, \beta) &= (\alpha_1, \alpha_2) \overset{\wedge}{\wedge} (\beta_1, \beta_2) \end{aligned}$$

Example


i_1
$a|(0.7, 0.2)$
w
i_2
$b|(0.4, 0.2)$
v
(i_1, i_2)
$(a, \bot)|(0.7, 0.2)$
(w, i_2)
$(\bot, b)|(0.4, 0.6)$
$(a, b)|(0.4, 0.2)$
$(\bot, b)|(0.4, 0.6)$
(i_1, v)
(w, v)
$(a, \bot)|(0.7, 0.2)$

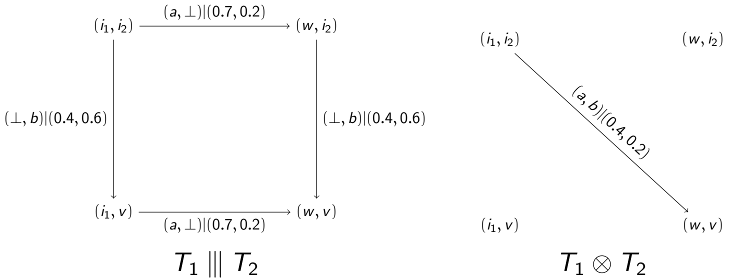
New PLTS from old

Interleaving: $T_1 \parallel\!| \, T_2 = (T_1 \times T_2) \restriction \lambda$

with the inclusion $\lambda : \{(a, \perp) \mid a \in Act_1\} \cup \{(\perp, b) \mid b \in Act_2\} \longrightarrow Act_1 \times_\perp Act_2$

Synchronous product: $T_1 \otimes T_2 = (T_1 \times T_2) \restriction \lambda$

taking $\{(a, b) \mid a \in Act_1 \text{ and } b \in Act_2\}$ as the domain of λ



New PLTS from old

Choice

$T_1 + T_2$ is the categorical coproduct in $Pt_{\mathbf{A}}$:

$$T_1 + T_2 = \langle W, (i_1, i_2), R \rangle$$

over $Act = Act_1 \cup Act_2$, where

- $W = (W_1 \times \{i_2\}) \cup (\{i_1\} \times W_2)$
- $R_a((w_1, w_2), (v_1, v_2)) = (\alpha, \beta)$ iff $(R_1)_a(w_1, v_1) = (\alpha, \beta)$ or $(R_2)_a(w_2, v_2) = (\alpha, \beta)$

$$(i_1, v) \xleftarrow{b|(0.4, 0.6)} (i_1, i_2) \xrightarrow{a|(0.7, 0.2)} (w, i_2)$$

New PLTS from old

Other Operators

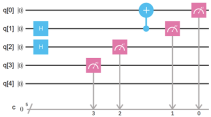
- Sequential composition as prefixing
- Functorial extension of operations from the underlying iMTL-algebra (e.g. to operate on weights)

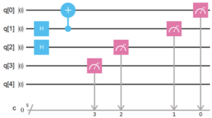
... leading to a sort of (paraconsistent) process algebra

An application: Modelling quantum circuits

- Transitions are labelled by the tensor of the relevant gates $O_1 \otimes \cdots \otimes O_m$ and weighted
- ... positively by the maximum of the longest qubit coherence times
- ... negatively by the minimum of the shortest qubit decoherence times, both reduced of the gate execution time.

An application: Modelling quantum circuits





$$s_1 \xrightarrow{(H_1 \otimes H_2, 1, 0)} s_2 \xrightarrow{(M_3, 0.99, 0.31)} s_3 \xrightarrow{(M_2, 0.98, 0.32)} s_4 \xrightarrow{(CX_{0,1}, 0.58, 0.72)} s_5 \xrightarrow{(M_1, 0.96, 0.33)} s_6 \xrightarrow{(M_0, 0.94, 0.40)} s_7$$

$$r_1 \xrightarrow{(H_1 \otimes H_2, 1, 0)} r_2 \xrightarrow{(CX_{0,1}, 0.6, 0.7)} r_3 \xrightarrow{(M_3, 0.98, 32)} r_4 \xrightarrow{(M_2, 0.97, 0.33)} r_5 \xrightarrow{(M_1, 0.96, 0.34)} r_6 \xrightarrow{(M_0, 0.94, 0.40)} r_7$$

$$\langle [*,*,*,*,*,*], 0.58, 0.72 \rangle \leq_c \langle [*,*,*,*,*,*], 0.6, 0.7 \rangle$$

Circuit optimization order

Weighted trace of a trace s

$$tw(s) = \langle \pi_1^*, \bigsqcap(\pi_2^*), \bigsqcup(\pi_3^*)\rangle\,(s)$$

Circuit optimization order

$\langle [a_1, a_2, ..., a_n], \alpha, \beta\rangle \;\leq_c\; \langle [b_1, b_2, ..., b_m], \gamma, \delta\rangle$ if

- sequence $[b_1, b_2, ..., b_m]$ is a prefix of $[a_1, a_2, ..., a_n]$,
- and $(\alpha, \beta) \preceq (\gamma, \delta)$

The logic $ML(\mathbf{A})$

Given an iMTL-algebra $\boldsymbol{A}$, a set of proposition symbols Prop and a set of action symbols Act:

$$\varphi \coloneqq \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi$$

where $p \in \mathrm{Prop}$ and $a \in Act$.
Abbreviations:

- $\top = \neg\bot$
- $\varphi \vee \varphi' = \neg(\neg\varphi \wedge \neg\varphi')$
- $\varphi \triangleright \varphi' = \neg\varphi \vee \varphi' = \neg(\varphi \wedge \neg\varphi')$
- $[a]\varphi = \neg\langle a \rangle \neg\varphi$.

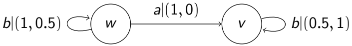
The logic $ML(\mathbf{A})$

Satisfaction: $\models : M \times Sen(\mathrm{Prop}, Act) \to A \times A$

$$(M \models \varphi) = \bigwedge_{w \in W} (M, w \models \varphi)$$

- $(M, w \models \bot) = (0, 1)$
- $(M, w \models p) = V(w, p)$
- $(M, w \models \neg\varphi) = \bar{\neg}(M, w \models \varphi)$
- $(M, w \models \varphi \wedge \varphi') = (M, w \models \varphi) \bigwedge (M, w \models \varphi')$
- $(M, w \models \langle a \rangle \varphi) = \bigvee_{v \in W} \left(R_a(w, v) \otimes (M, v \models \varphi) \right)$

Example over the Gödel algebra



$V(w,p) = (1,1)$, $V(w,q) = V(v,p) = (0,0.5)$, $V(w,r) = V(v,r) = (0.5,0.5)$, $V(v,q) = (0,0)$.

$$
\begin{aligned}
(M,w \models r \rhd (p \vee q)) &= (M,w \models \neg r \vee (p \vee q)) \\
&= (M,w \models \neg(\neg\neg r \wedge \neg(p \vee q))) \\
&= \overline{\neg}\Big(M,w \models r \wedge \neg(p \vee q)\Big) \\
&= \overline{\neg}\Big((M,w \models r) \overset{\wedge}{\wedge} (M,w \models \neg(p \vee q))\Big) \\
&= \overline{\neg}\Big(V(w,r) \overset{\wedge}{\wedge} /\!\!/ (M,w \models p \vee q)\Big) \\
&= \overline{\neg}\Big(V(w,r) \overset{\wedge}{\wedge} (M,w \models \neg p) \overset{\wedge}{\wedge} (M,w \models \neg q)\Big) \\
&= \overline{\neg}\Big(V(w,r) \overset{\wedge}{\wedge} \overline{\neg}V(w,p) \overset{\wedge}{\wedge} \overline{\neg}V(w,q)\Big) \\
&= \overline{\neg}\Big((0.5,0.5) \overset{\wedge}{\wedge} (1,1) \overset{\wedge}{\wedge} (0.5,0)\Big) \\
&= \overline{\neg}(0.5 \wedge 1 \wedge 0.5, 0.5 \vee 1 \vee 0) = \overline{\neg}(0.5,1) \\
&= (1,0.5)
\end{aligned}
$$

Simulation, Bisimulation & Invariance

The crisp case

- $S \subseteq W_1 \times W_2$ is a simulation if, for any $(w_1, w_2) \in S$, $a \in Act$,

  $R_a(w_1, w_1') = (\alpha, \beta)$ then
  $\exists_{w_2' \in W_2} \exists_{\gamma, \delta \in A}.\ R(w_2, w_2') = (\delta, \gamma)$ and $(w_1', w_2') \in S$ and $(\alpha, \beta) \preccurlyeq (\delta, \gamma) \rangle$

  For all $p \in \mathrm{Prop}$ and $(w, v) \in S$, $V_1(w, p) \preccurlyeq V_2(v, p)$

- $\langle w_1, w_2 \rangle \in S$ entails $(w_1 \models \varphi) \preccurlyeq (w_2 \models \varphi)$ for $\varphi \in \mathrm{Fm}^{+\Diamond}$, the positive fragment of $\mathrm{L}(\mathbf{A})$
- Bisimilarity ensures modal invariance, but is too strong; other equivalences are useful in practice.

Simulation, Bisimulation & Invariance

The graded case

- Are themselves positive,negative-weighted relations
- $B : W \times W' \to A \times A$ is a graded bisimulation if

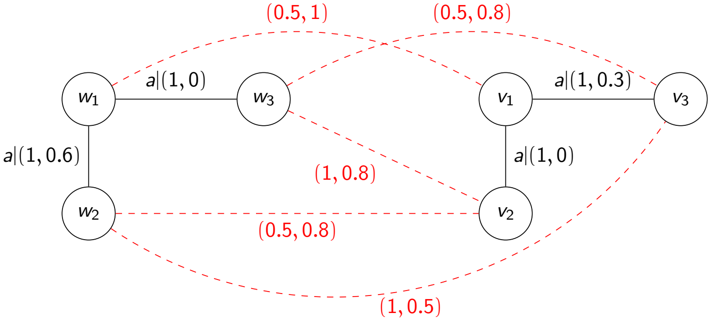
$$B(w, w') \preccurlyeq (V(w, p) \Leftrightarrow V'(w', p))$$
$$\exists v' \in W' \; (B(w, w') \otimes R_a(w, v)) \preccurlyeq (R'_a(w', v') \otimes B(v, v'))$$
$$\exists v \in W \; (B(w, w') \otimes R'_a(w', v')) \preccurlyeq (R_a(w, v) \otimes B(v, v'))$$

- Invariance result for graded bisimulation:

$$B(w, w') \preccurlyeq \Big((M, w \models \varphi) \Leftrightarrow (M', w' \models \varphi)\Big)$$

- The crisp case is not a particular case

Simulation, Bisimulation & Invariance



Concluding

A lot remains to be done ...

- Development of the logic's proof-theoretic perspective, as a major step towards (semi-)automatic support to reasoning about such complex, often weird, but actually quite common phenomena
- Dynamic logics for paraconsistent programming
- with a particular instance for hybrid and noisy quantum programs

Concluding

Where to read more ...

- Cruz, Madeira & Barbosa: A logic for paraconsistent transition systems
  Non-Classical Logics: Theory and Applications, 2022.
- Cruz, Madeira & Barbosa: Paraconsistent transition systems
  LSFA'22 (Logical and Semantic Frameworks with Applications) , 2022.
- Cunha, Madeira & Barbosa: Structured specification of paraconsistent transition systems
  FSEN'23 (Fundamentals of Software Engineering), 2023.
- Barbosa & Madeira: Capturing qubit decoherence through paraconsistent transition systems
  Engineering of Quantum Programming Workshop, IEEE 2023
- Cunha, Madeira & Barbosa: Paraconsistent transition structures: compositional principles and a modal logic
  Math. Struc. Comp. Sci., Elsevier (in print) 2023

Concluding: Paraconsistency everywhere ...

When the hills are all flat,
The rivers are all dry.

When it thunders in winter,
When it snows in summer.

When heaven and earth mingle,

Not till then will I part from you.

Yuefu folk poems, Han Dynasty