Going quantitative in software modeling

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Motivation

Software modeling

Simplicity + elegance = **effectiveness** (Dijkstra)

Alloy — *writing less to say more* :-)

However: qualitative features simpler to model than quantitative ones

"Quantitative **abstraction**"?

"Scalable modeling": the *"keep definition, change category"* lemma.

Starting point — what is **modeling language**, after all?

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Starting point — what is **modeling language**, after all?

Category

Abstract language made of arrows which (may) compose with each other, and such that

(a) **associativity**

$$c \cdot (b \cdot a) = (c \cdot b) \cdot a \qquad (1)$$

holds.

$$\begin{tikzcd}
q & \\
k \arrow[u, "c"'] & m \arrow[l, "b"'] \arrow[ul, "c\cdot b"'] \\
& n \arrow[u, "a"'] \arrow[ul, "b\cdot a"]
\end{tikzcd}$$

(b) every **object** a has an **identity** such that:

$$1 \cdot a = a \cdot 1 = a \qquad (2)$$

$$\begin{tikzcd}
m & m \arrow[l, "1"'] \\
n \arrow[u, "a"] & n \arrow[l, "1"] \arrow[u, "a"'] \arrow[ul, "a"']
\end{tikzcd}$$

Thus, arrows form a **monoid**.

Enriched categories

Arrows can be added

$$n \xrightarrow{a} m \qquad n \xrightarrow{b} m \qquad n \xrightarrow{a+b} m$$

and can be multiplied

$$n \xrightarrow{a} m \qquad n \xrightarrow{b} m \qquad n \xrightarrow{a\times b} m$$

such that, under $(\cdot)$, $\times$ and $+$, arrows form two **semirings**:

$$
\begin{array}{ll}
a + (b + c) = (a + b) + c & a + 0 = a = 0 + a \\
a \times (b \times c) = (a \times b) \times c & a \times \top = a = \top \times a \\
a + b = b + a & \\
a \times (b + c) = a \times b + a \times c & a \times 0 = 0 = 0 \times a \\
a \cdot (b + c) = a \cdot b + a \cdot c & a \cdot 0 = 0 = 0 \cdot a
\end{array}
$$

"Dagger" categories

Further structure — for every arrow $k \xrightarrow{a} q$ there exists an arrow $k \xleftarrow{a^\circ} q$, the **converse** of a, such that:

$$
\begin{aligned}
(a^\circ)^\circ &= a \\
(a \cdot b)^\circ &= b^\circ \cdot a^\circ \\
(a + b)^\circ &= a^\circ + b^\circ \\
(a \times b)^\circ &= a^\circ \times b^\circ
\end{aligned}
$$

NB: "dagger" because a° often written as $a^\dagger$.

Famous **counter**-example: category of sets and functions.

Idempotency

Additive operator $+$ makes a difference.

$+$-**idempotency**: wherever $a + a = a$ holds for all a, then

$$a \leqslant b \quad \stackrel{\mathrm{def}}{=} \quad a + b = b \tag{3}$$

is a partial order.

Clearly, $0 \leqslant a$ for all a and $(+)$ is the *lub* with respect to $\leqslant$:

$$a + b \leqslant c \quad \equiv \quad a \leqslant c \wedge b \leqslant c \tag{4}$$

NB: $c := a + b$ in (4) means $a + b$ is upper bound; $\Leftarrow$ means it is the **least** upper bound (*lub*).

Relational algebra is an example of such idempotency (next slide).

Binary relations

The algebra of **binary relations** is a well known example of such enriched categories:

Categorial	**Binary relations**	**Description**
$x \cdot y$	$R \cdot S$	composition
$x + y$	$R \cup S$	union
$x \times y$	$R \cap S$	intersection
0	$\perp$	empty relation
1	id	identity relation
$\top$	$\top$	top relation
x°	R°	converse relation
$x \leqslant y$	$R \subseteq S$	inclusion

$\cup$-**idempotency** brings about the $R \subseteq S$ partial order, thus enabling recursion, iteration etc. — but it hinders implicit expression of **quantities** (cf. numbers in Alloy).

Matrices

In case addition is **not** idempotent — eg. $x + x = 2\,x$ — we get a typed **linear algebra** of matrices ("as arrows"):

Categorial	Matrices	Description
$x \cdot y$	$M \cdot N$	MMM
$x + y$	$M + N$	pointwise addition
$x \times y$	$M \times N$	Hadamard product
0	$\bot$	everywhere-0 matrix
1	id	identity matrix
$\top$	$\top$	everywhere-1 matrix
x°	M°	transpose matrix

$\{0, 1\}$-valued (Boolean) matrices represent binary relations, where

$$M \cap N = M \times N$$
$$M \cup N = M + N - M \times N.$$

(So the $+$-semiring must be a **ring**.) By default, in this talk we assume $\mathbb{Z}$-valued matrices.

Functions

Functions are Boolean matrices (relations) such that $! \cdot f = !$, where $k \xrightarrow{\ !\ } 1 = \top$. ($!$ is itself a function; $1 \xrightarrow{\ !\ } 1 = id$.)

Functions enjoy quite a number of properties, in particular, for f and g functions,

$$
\begin{aligned}
y\,(g^\circ \cdot M \cdot f)\,x &= (g\,y)\,M\,(f\,x) && (5)\\
y\,(f \cdot M)\,x &= \langle \textstyle\sum z : y = f\,z : z\,M\,x \rangle && (6)\\
y\,(M \cdot f^\circ)\,x &= \langle \textstyle\sum z : x = f\,z : y\,M\,z \rangle && (7)
\end{aligned}
$$

For relations, similar laws hold just by replacing $\sum z$ by $\exists\, z$.

In the sequel, we shall denote by $\mathbb{R}$ — resp. $\mathbb{M}$ — the category of binary **relations** — resp. $\mathbb{Z}$-valued **matrices**.

Abstract model

In $\mathbb{R}$, to begin with:

$$
\begin{tikzcd}
 & Keyword & \\
Paper \arrow[ur, "K"] \arrow[r, "C"] & Paper \arrow[r, "A"] \arrow[d, "m"] & Author \\
 & Medium &
\end{tikzcd}
\qquad (8)
$$

where

- $c' \; C \; c$ means c' is **cited** by c or c cites c'
- $k \; K \; p$ means that paper p has **keyword** k
- $m \; p$ is *the* **publication** medium of paper p (a function)
- $a \; A \; p$ means a is among the **authors** of paper p.

Alloy

$$
\begin{array}{l}
\textbf{sig}\ \mathit{Paper}\ \{\\
\quad C : \textbf{set}\ \mathit{Paper},\\
\quad K : \textbf{set}\ \mathit{Keyword},\\
\quad A : \textbf{set}\ \mathit{Author},\\
\quad m : \textbf{one}\ \mathit{Medium}\\
\}
\end{array}
$$

Papers cannot cite themselves: $C \subseteq \neg id$, that is

$$\textbf{fact}\ \{\textbf{no}\ \ C\ \&\ \mathit{iden}\}$$

since $\neg R = R \Rightarrow \bot$ and implication is defined by GC

$$X \cap Y \subseteq Z \Leftrightarrow X \subseteq Y \Rightarrow Z. \tag{9}$$

Triangular patterns

In category $\mathbb{R}$:

$$
\begin{tikzcd}
 & Keyword \\
Paper \arrow[ur, "K"] \arrow[ur, bend left, "R"] \arrow[r, "C"] & Paper \arrow[u, "K"] \arrow[u, bend right, "S"']
\end{tikzcd}
\qquad
\left\{
\begin{array}{l}
R = K \cap K \cdot C \\
S = K \cap K \cdot C^{\circ}
\end{array}
\right.
$$

R is not particularly interesting.

But S is so,

$$k \; S \; p \Leftrightarrow k \; K \; p \wedge \langle \exists \; q \; : \; p \; C \; q : \; k \; K \; q \rangle$$

meaning: *paper* p *is cited by at least another (btw different) paper* q *"in the same area" (keyword* k*).*

Triangular patterns (composition)

$$\begin{tikzcd} & Keyword & \\ Paper \arrow[ur, "K"] \arrow[r, "C"'] & Paper \arrow[u, "S"] \arrow[r, "A"'] & Author \arrow[ul, "Q"'] \end{tikzcd}$$

$$\begin{cases} S = K \cap K \cdot C^\circ \\ Q = S \cdot A^\circ \end{cases}$$

Then

$$Keyword \xleftarrow{Q} Author \ = S \cdot A^\circ$$

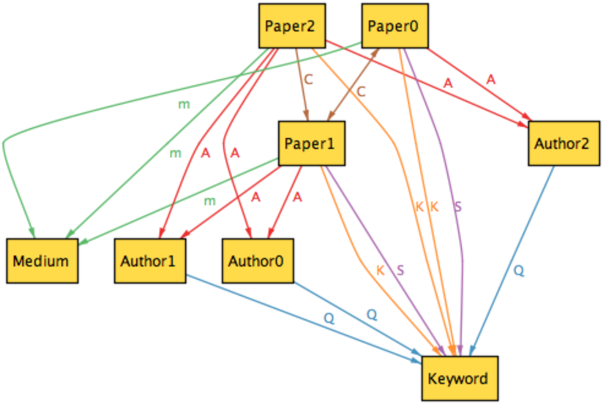
is such that

$$k \ Q \ a = \langle \exists \ p \ : \ a \ A \ p : \ k \ S \ p \rangle$$

telling which authors have cited papers in particular areas (keywords).

Alloy

We can do model analysis...



... but no bibliometrics! Why? Idempotency!

Alloy

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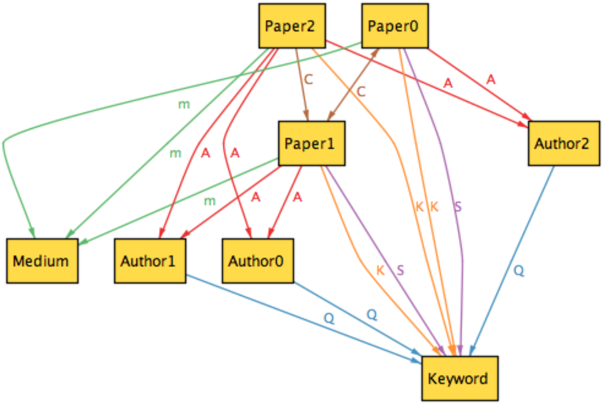


... but no biblio<u>metrics</u>! Why? Idempotency!

Keep definition!

Shall we add **quantitative** information to the model?

No! *Recall scalable modeling: "keep definition, change category".*

It suffices to interpret the **same** (abstract) model in category $\mathbb{M}$ — e.g. pattern

$$Keyword \xleftarrow{S} Paper \;= K \times (K \cdot C^\circ)$$

will now **count** how many papers cite a given one, all within the same area:

$$\begin{array}{l} k\ S\ p = \\ \quad \textbf{if}\ (k\ K\ p)\ \textbf{then}\ \langle \sum\ q\ :\ p\ C\ q \wedge k\ K\ q :\ 1 \rangle\ \textbf{else}\ 0 \end{array}$$

Keep definition!

In $\mathbb{M}$, the arrows in

$$\begin{tikzcd}
& Keyword & \\
Paper \arrow[ur, "K"] \arrow[r, "C"] & Paper \arrow[u, "S"] \arrow[r, "A"] & Author \arrow[ul, "Q"']
\end{tikzcd}
\qquad
\left\{ \begin{array}{l} S = K \times (K \cdot C) \\ Q = S \cdot A^{\circ} \end{array} \right.$$

are still relations but, as the category changed,

$$Keyword \xleftarrow{Q} Author \;\; = S \cdot A^{\circ}$$

is such that

$$k \; Q \; a = \langle \textstyle\sum \; p \; : \; a \; A \; p : \; k \; S \; p \rangle \tag{10}$$

— it gives, for each author, her/his histogram of citations per keyword, within the same area.

Percentiles

Pushing further, $\mathbb{M}$ can be enriched so that $\times$ forms a **group**, bringing division in:

$$Keyword \xleftarrow{Z} Author \;=\; \frac{S \cdot A^\circ}{S \cdot \top} \tag{11}$$

This makes such histograms relative to the **grand total** of citations in each area (keyword) k:

$$k\ Z\ a = \frac{\langle \sum p\ :\ a\ A\ p :\ k\ S\ p \rangle}{\langle \sum q\ ::\ k\ S\ q \rangle}$$

That is, $k\ Z\ a$ gives the percentile of author a when evaluated (with)in keyword (area) k.

Example: $k\ Z\ a = n\ 10^{-5}$ means that n-many citations among 10^5 citations in area k are of papers by a.

h-index?

Z metric better than h-index because it takes into accout the cardinality of each community (cf. keywords).

h-index harder to encode (is there a "ranking" semiring?)

(Thinking about this — too many frustrating committees!)

More triangular patterns (metaphors)

In $\mathbb{R}$, another triangular pattern is

$$
\begin{tikzcd}
T \arrow[dr, "g"'] & & V \arrow[ll, "\frac{f}{g} = g^{\circ} \cdot f"'] \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\qquad t \, \frac{f}{g} \, v \Leftrightarrow (g \; t) = (f \; v) \tag{12}
$$

— where f, g are functions — called a **metaphor**.

By (5), this has the **same** meaning in category $\mathbb{M}$.

Nice properties, recalling **rational numbers**, e.g.

$$
\frac{f}{id} = f \tag{13}
$$

$$
\left(\frac{f}{g}\right)^{\circ} = \frac{g}{f} \tag{14}
$$

and so on.

Rational matrices / relations

$\frac{f}{g}$	$a1$	$a2$	$a3$	$a4$	$a5$
b_1	0	0	1	0	1
b_2	1	0	0	0	0
$b3$	0	1	0	1	0
$b4$	0	1	0	1	0
$b5$	0	0	0	0	0

$$
f = \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}
$$

$$
g = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}
$$

Most specifications are rational relations / matrices, eg.

$$
Sort = \frac{bag}{bag} \times \frac{true}{ordered} \qquad \left(= \frac{bag \triangledown true}{bag \triangledown ordered}\right)
$$

where $(f \triangledown g)\ a = (f\ \ a, g\ \ a)$.

Quantitative invariants

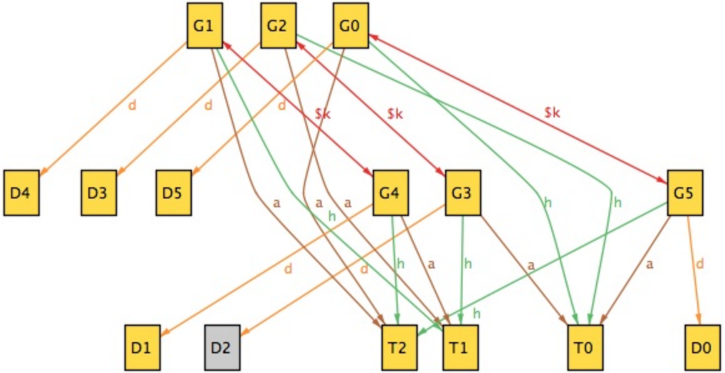
The teams (T) of a football league play games (G) at home (h) or away (a), and every game takes place in some date (d):

$$\begin{tikzcd} T & G \arrow[l, "h"'] \arrow[r, "a"] \arrow[d, "d"'] & T \\ & D & \end{tikzcd}$$

Invariantly,

- All teams play against each other exactly once but never against themselves.
- No team can play two games on the same date.

Three teams playing



Clearly, $k = \frac{a^\triangledown h}{h^\triangledown a}$ should be a bijection (cf. team swapping).

Quantitative invariants

All teams play again each other exactlty once but never against themselves, in $\mathbb{R}$:

$$h \cdot a^{\circ} = \top - id \tag{15}$$

meaning, for all teams t, t'

$$\langle \exists\, x \ :: \ t = h\, x \wedge t' = a\, x \rangle \Leftrightarrow t \neq t'$$

Exactly once? In $\mathbb{M}$ we write exactly the same as above,

$$h \cdot a^{\circ} = \top - id$$

capturing everything:

For all teams t, t',

$$\langle \sum x \ : \ t = h\, x \wedge t' = a\, x : \ 1 \rangle = \textbf{if } t = t' \textbf{ then } 0 \textbf{ else } 1$$

Quantitative invariants

No team can play two games on the same date, in $\mathbb{R}$:

$$d^{\circ} \cdot d \subseteq id \cup \neg(I^{\circ} \cdot I)$$

where $I = a \cup h$, $t \; I \; x$ meaning "team t is involved in game x".

That is, for all $x \not\equiv x'$,

$$\langle \exists \; t \; :: \; t \; I \; x \wedge t \; I \; x' \rangle \Rightarrow (d \; x) \not\equiv (d \; x')$$

Interestingly, in $\mathbb{M}$ this invariant is rendered much simpler,

$$d \cdot (a + h)^{\circ} \leqslant \top$$

cf.

$$\langle \forall \; y, t \; :: \; \langle \sum x \; : \; y = d \; x : \; t \; a \; x + t \; h \; x \rangle \rangle \leqslant 1$$

Quantitative invariants

Recall that $k = \frac{a \triangledown h}{h \triangledown a}$ should be a bijection.

Bijection = function (= deterministic + total) + injective + surjective

---

In $\mathbb{M}$:

$$! \cdot \frac{a \triangledown h}{h \triangledown a} = !$$

$$! \cdot \frac{h \triangledown a}{a \triangledown h} = !$$

It all has to do with **totals** — **counting** how many 1s the (Boolean) matrices have per column /row !

Wrapping up

Main idea:

"Scalable modeling": the "keep definition, change category" lemma.

In the previous TRUST workshop I played the same game with another category, that of **Markov chains**.

Questions:

- What is the best path towards **quantitative abstraction**?
- Some pointfree statements simpler if idempotency is removed
- What would it mean for Alloy to drop $+$-idempotency? (cf. SMT backend)

Annex

Annex

Pointwise details of (10):

$$
\begin{aligned}
k\ Q\ a &= \langle \sum\ p\ :\ a\ A\ p :\ k\ S\ p \rangle \\
&= \langle \sum\ p\ :\ a\ A\ p \wedge k\ K\ p :\ \langle \sum\ q\ :\ p\ C\ q \wedge k\ K\ q :\ 1 \rangle \rangle \\
&= \langle \sum\ p, q\ :\ a\ A\ p \wedge k\ K\ p \wedge p\ C\ q \wedge k\ K\ q :\ 1 \rangle
\end{aligned}
$$